Public Key Cryptography and RSA

ITC 3093 Principles of Computer Security

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Every Egyptian received two names, which were known respectively as the true name and the good name, or the great name and the little name; and while the good or little name was made public, the true or great name appears to have been carefully concealed.

—*The Golden Bough, Sir James George Frazer*

Private-Key Cryptography

- traditional **private/secret/single key** cryptography uses **one** key
- shared by both sender and receiver
- if this key is disclosed communications are compromised
- also is **symmetric**, parties are equal
- hence does not protect sender from receiver forging a message & claiming is sent by sender

Public-Key Cryptography

- probably most significant advance in the 3000 year history of cryptography
- uses **two** keys – a public & a private key
- **asymmetric** since parties are **not** equal
- uses clever application of number theoretic concepts to function
- complements **rather than** replaces private key crypto

Why Public-Key Cryptography?

- developed to address two key issues:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- public invention due to Whitfield Diffie & Martin Hellman at Stanford Uni in 1976
  - known earlier in classified community

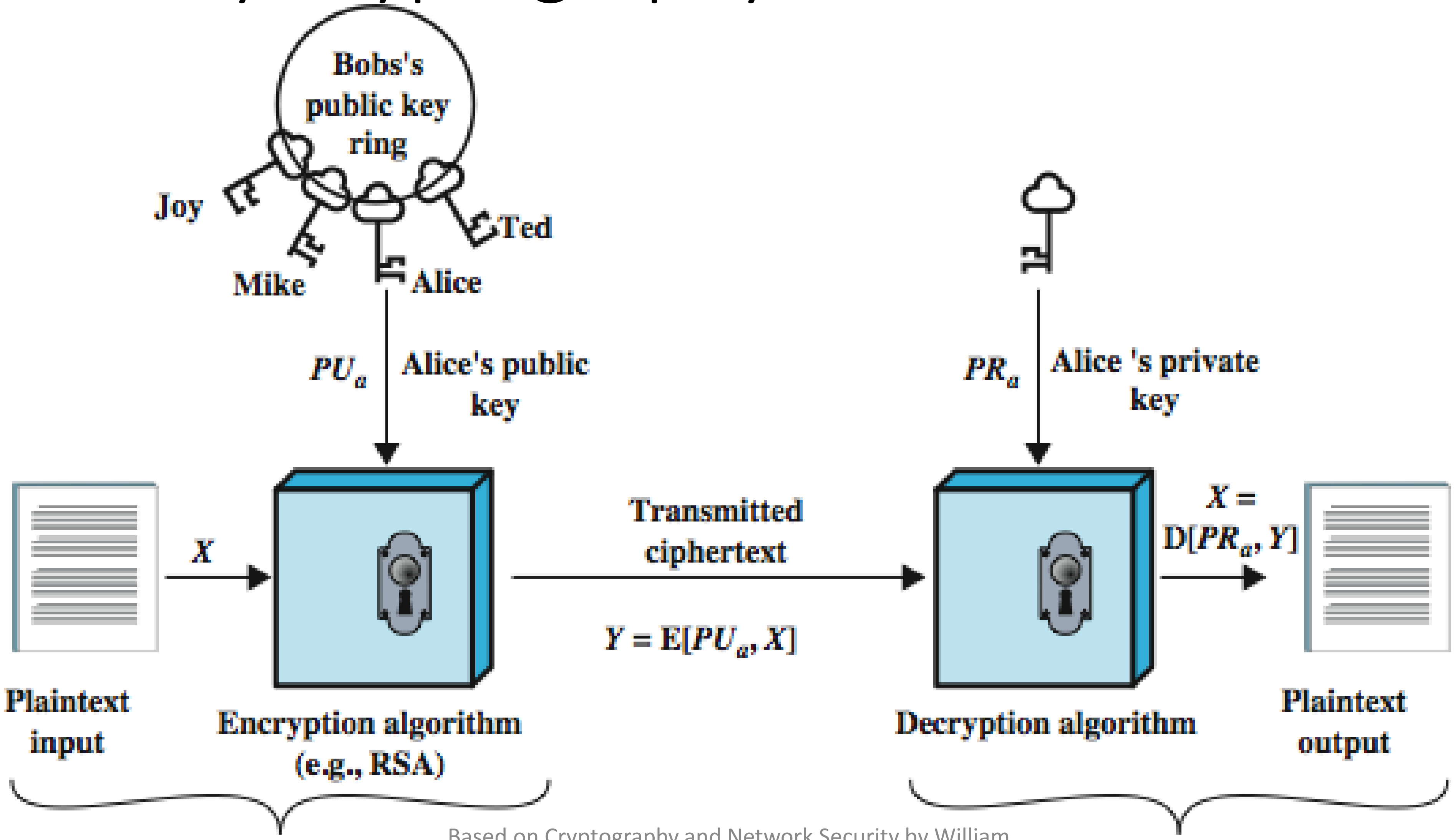
Public-Key Cryptography

- **public-key/two-key/asymmetric** cryptography involves the use of **two** keys:
  - a **public-key**, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a related **private-key**, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**
- **infeasible to determine private key from public**
- is **asymmetric** because
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

Public-Key Cryptography

- Consider the following analogy using padlocked boxes:
- Traditional schemes involve the sender putting a message in a box and locking it, sending that to the receiver, and somehow securely also sending them the key to unlock the box.
- The radical advance in public key schemes was to turn this around, the receiver sends an **unlocked box** (their public key) to the sender.
- Sender puts the message in the box and locks it (easy - and having locked it cannot get at the message), and sends the locked box to the receiver who can unlock it (also easy), having the (private) key.
- An attacker would have to pick the lock on the box (hard).

Public-Key Cryptography

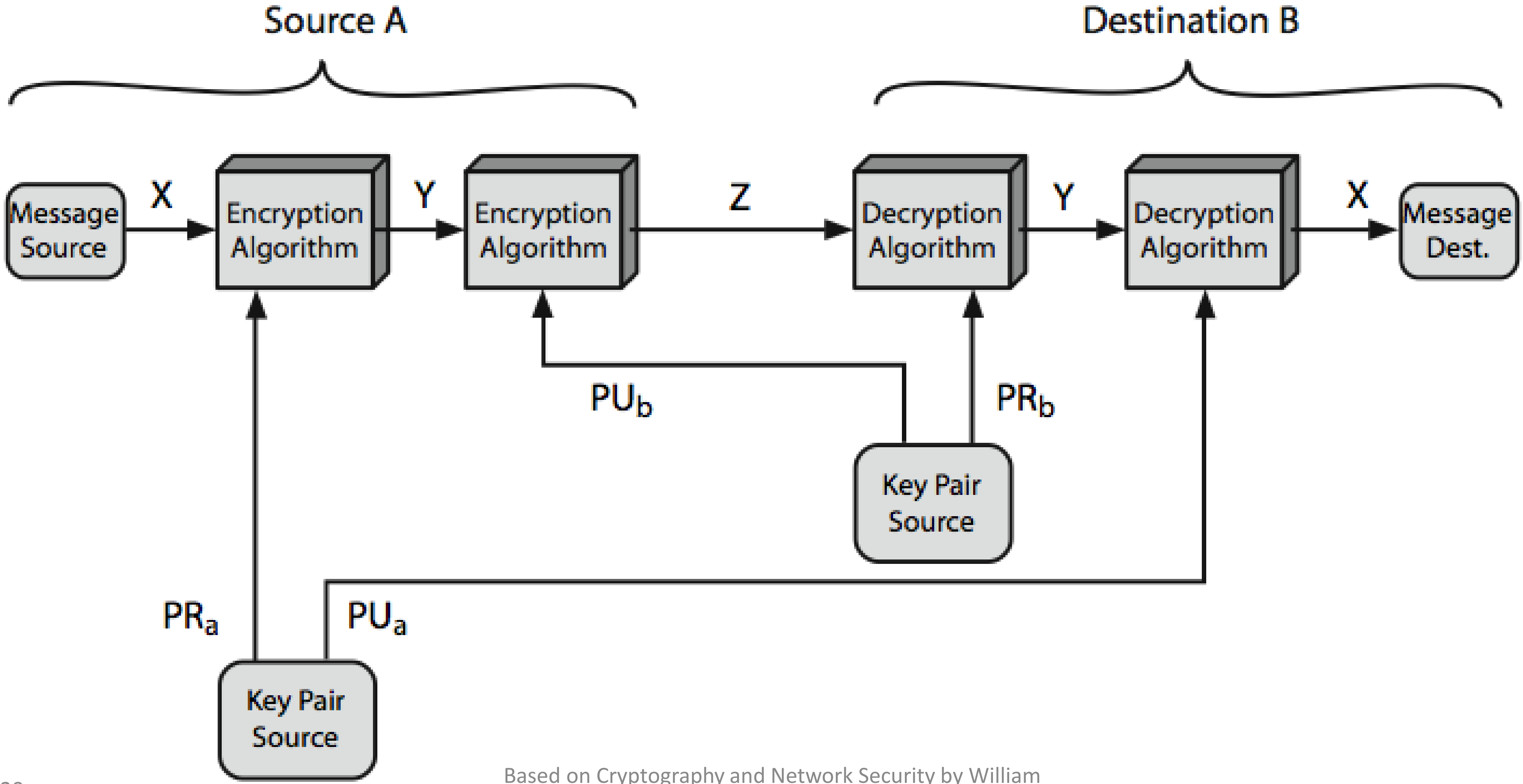


(a) Encryption with public key

Symmetric vs Public-Key

Conventional Encryption	Public-Key Encryption
Needed to Work: 1. The same algorithm with the same key is used for encryption and decryption. 2. The sender and receiver must share the algorithm and the key. *Needed for Security:* 1. The key must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus samples of ciphertext must be insufficient to determine the key.	*Needed to Work:* 1. One algorithm is used for encryption and decryption with a pair of keys, one for encryption and one for decryption. 2. The sender and receiver must each have one of the matched pair of keys (not the same one). *Needed for Security:* 1. One of the two keys must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus one of the keys plus samples of ciphertext must be insufficient to determine the other key.

Public-Key Cryptosystems



Public-Key Applications

- can classify uses into 3 categories:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

Public-Key Requirements

- Public-Key algorithms rely on two keys where:
  - it is computationally infeasible to find decryption key knowing only algorithm & encryption key
  - it is computationally easy to en/decrypt messages when the relevant (en/decrypt) key is known
  - either of the two related keys can be used for encryption, with the other used for decryption (for some algorithms)
- these are formidable requirements which only a few algorithms have satisfied

Public-Key Requirements

- need a trapdoor one-way function
- one-way function has
  - $Y = f(X)$ easy
  - $X = f^{-1}(Y)$ infeasible
- a trap-door one-way function has
  - $Y = f_k(X)$ easy, if k and X are known
  - $X = f_k^{-1}(Y)$ easy, if k and Y are known
  - $X = f_k^{-1}(Y)$ infeasible, if Y known but k not known
- a practical public-key scheme depends on a suitable trap-door one-way function

Security of Public Key Schemes

- like private key schemes brute force **exhaustive search** attack is always theoretically possible
- but keys used are too large (>512bits)
- security relies on a **large enough** difference in difficulty between **easy** (en/decrypt) and **hard** (cryptanalyse) problems
- more generally the **hard** problem is known, but is made hard enough to be impractical to break
- requires the use of **very large numbers**
- hence is **slow** compared to private key schemes

One way functions

a function that is “easy” to compute and “difficult” to reverse.

OWF: Multiplying two primes

- It is easy to take two prime numbers and multiply them together.
- If they are fairly small we can do this in our heads, on a piece of paper, or on a calculator.
- As they get bigger and bigger it is fairly easy to write a computer program to compute the product.
- Multiplication runs in polynomial time.
- Multiplication of two primes is easy.

OWF: Multiplying two primes

To factor:	Comments
15	
143	
6887	
31897	
A 600 digit number	
A 600 digit even number	

OWF: Multiplying two primes

- Multiplication of two prime numbers is **believed** to be a one-way function.
- We say **believed** because nobody has been able to **prove** that it is hard to factorise.
- Maybe one day someone will find a way of factorising efficiently.
- What will happen if someone does find an efficient way of factorising?

OWF: Modular exponentiation

- The process of **exponentiation** just means raising numbers to a power.
- Raising **a** to the power **b**, normally denoted a^b just means multiplying **a** by itself **b** times. In other words:

$$a^b = a \times a \times a \times \ldots \times a$$

- **Modular exponentiation** means computing a^b modulo some other number **n**. We tend to write this as

$$a^b \bmod n.$$

- Modular exponentiation is “easy”.

OWF: Modular exponentiation

- However, given **a**, **b**, and **a^b mod n** (when **n** is prime), calculating **b** is regarded by mathematicians as a hard problem.
- This difficult problem is often referred to as the **discrete logarithm problem**.
- In other words, given a number **a** and a prime number **n**, the function

$$f(b) = a^b \bmod n$$

- is believed to be a one-way function.

OWF: Modular square roots

- What is the square root of 1369?
  - Propose a technique for finding the square root of 1369 that will generalise to any integer.

- What is the square root of 56 module 101?
  - Let's try 40…
  - Let's try 30…

Suitable OWFs

- We have seen that the encryption process of a public key cipher system requires a one way function.
- Is every one way function suitable for implementation as the encryption process of a public key cipher system?

RSA

RSA

- by Rivest, Shamir & Adleman of MIT in 1977
- best known & widely used public-key scheme
- based on exponentiation in a finite (Galois) field over integers modulo a prime
    - nb. exponentiation takes $O((\log n)^3)$ operations (easy)
- uses large integers (eg. 1024 bits)
- security due to cost of factoring large numbers
    - nb. factorization takes $O(e^{\log n \log \log n})$ operations (hard)

RSA En/decryption

- to encrypt a message M the sender:
  - obtains **public key** of recipient $PU=\{e,n\}$
  - computes: $C = M^e \bmod n$, where $0 \leq M < n$
- to decrypt the ciphertext C the owner:
  - uses their private key $PR=\{d,n\}$
  - computes: $M = C^d \bmod n$
- both sender and receiver must know the value of n.
- sender knows the value of e, and only the receiver knows the value of d.

RSA Example - En/Decryption

- sample RSA encryption/decryption is:
- given message $M = 88$ (nb. $88<187$)
- encryption:
  - $C = 88^7 \bmod 187 = 11$
- decryption:
  - $M = 11^{23} \bmod 187 = 88$

RSA Key Generation

- users of RSA must:
  - determine two primes at random - `p, q`
  - select either `e` or `d` and compute the other
- primes `p,q` must not be easily derived from modulus `n=p.q`
  - means must be sufficiently large
  - typically guess and use probabilistic test
- exponents `e`, `d` are inverses, so use inverse algorithm to compute the other

RSA Security

- possible approaches to attacking RSA are:
  - brute force key search - infeasible given size of numbers
  - mathematical attacks - based on difficulty of computing $\phi(n)$, by factoring modulus n
  - timing attacks - on running of decryption
  - chosen ciphertext attacks - given properties of RSA

Factoring Problem

- mathematical approach takes 3 forms:
  - factor n=p.q, hence compute ø(n) and then d
  - determine ø(n) directly and compute d
  - find d directly
- currently believe all equivalent to factoring
  - have seen slow improvements over the years
    - as of May-05 best is 200 decimal digits (663) bit with LS
  - biggest improvement comes from improved algorithm
    - cf QS to GHFS to LS
  - currently assume 1024-2048 bit RSA is secure
    - ensure p, q of similar size and matching other constraints

Summary

- Public key systems replace the problem of distributing symmetric keys with one of authenticating public keys
- Public key encryption algorithms need to be trapdoor one-way functions
- RSA is a public key encryption algorithm whose security is believed to be based on the problem of factoring large numbers